SMALL-SCALE FLUCTUATIONS OF RELIC RADIATION*

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(Received 11 September, 1969)

Abstract. Perturbations of the matter density in a homogeneous and isotropic cosmological model which leads to the formation of galaxies should, at later stages of evolution, cause spatial fluctuations of relic radiation. Silk assumed that an adiabatic connection existed between the density perturbations at the moment of recombination of the initial plasma and fluctuations of the observed temperature of radiation $\delta T/T = \delta\varrho_m/3\varrho_m$. It is shown in this article that such a simple connection is not applicable due to:

(1) The long time of recombination;

(2) The fact that when regions with $M < 10^{15}\, M_\odot$ become transparent for radiation, the optical depth to the observer is still large due to Thompson scattering;

(3) The spasmodic increase of $\delta\varrho_m/\varrho_m$ in recombination.

As a result the expected temperature fluctuations of relic radiation should be smaller than adiabatic fluctuations. In this article the value of $\delta T/T$ arising from scattering of radiation on moving electrons is calculated; the velocity field is generated by adiabatic or entropy density perturbations. Fluctuations of the relic radiation due to secondary heating of the intergalactic gas are also estimated. A detailed investigation of the spectrum of fluctuations may, in principle, lead to an understanding of the nature of initial density perturbations since a distinct periodic dependence of the spectral density of perturbations on wavelength (mass) is peculiar to adiabatic perturbations. Practical observations are quite difficult due to the smallness of the effects and the presence of fluctuations connected with discrete sources of radio emission.

1. Introduction

In the contemporary 'big-bang' model of the Universe it is hypothesized that in the distant past, before recombination of the initial plasma at times corresponding to a red shift $z \sim 1000$, there were no galaxies and the origin of galaxies is connected with insignificant deviations from strict homogeneity existing in that period. In the first approximation it can be considered that after recombination of protons and electrons 'matter' – neutral atoms – do not interact with radiation and relic radiation (having at present an average temperature of 2.7 K) immediately gives us information about conditions for $z \sim 1000$. In particular, the dependence of deviations of temperature on the direction of observation now being performed from the Earth characterizes the dependence of physical values, i.e. deviations of density, on the spatial coordinates at an earlier stage. These deviations grow in the future (after recombination) due to gravitation instability. For the moment of formation of separate objects it is reasonable to take the time of origin of regions with densities at least twice the average density, i.e. $\delta\varrho/\varrho \sim 1$. It is assumed that this occurred relatively recently (on a logarithmic time scale) at $z \sim 2 \approx 10$. In this case an estimate for the perturbation at the moment of recombination gives $\delta\varrho/\varrho \sim 10^{-2}$–$10^{-3}$, i.e., it is possible to speak about

* Translated from the Russian by D. F. Smith.

Astrophysics and Space Science 7 (1970) 3–19.

small perturbations. Not only the perturbations which lead to formation of galaxies, but the whole spectrum of perturbations is of interest for the characteristics of the initial inhomogeneity.

The natures of the perturbations may be qualitatively divided:

(1) Density perturbations of nuclei and electrons ϱ_m on a background of a constant density of quanta ϱ_r (so-called entropy perturbations).

(2) Compression and rarefaction waves of the plasma as a whole with simultaneous changes of ϱ_m and ϱ_r (adiabatic perturbations).

(3) Turbulent motions of the plasma.

(4) Chaotic magnetic fields and perhaps other types of perturbations.

Different types of perturbations evolve differently at plasma periods ($z > 1400$) and give different predictions concerning the formation of galaxies and fluctuations of relic radiation. Silk (1967a, b; 1968) first made quantitative predictions concerning adiabatic perturbations. His results were obtained on the assumption that recombination of the initial plasma occurs quite suddenly for a definite z_r. Earlier in time, for $z > z_r$, perfect adiabaticity is assumed so that

$$\frac{\delta\varrho_m}{\varrho_m} = 3\,\frac{\delta T}{T}, \quad \frac{\delta\varrho_r}{\varrho_r} = 4\,\frac{\delta T}{T}.$$

Later, for $z < z_r$, matter is completely transparent and an observer measures $\delta T/T$ reached at the moment $z = z_r$ directly. On the other hand, due to gravitational instability $\delta\varrho/\varrho$ subsequently grows proportional to $(1+z)^{-1}$, so that

$$\left(\frac{\delta\varrho}{\varrho}\right)_{z_r} = \frac{1+z_0}{z_r}\left(\frac{\delta\varrho}{\varrho}\right)_0 = \frac{1+z_0}{z_r}$$

(for a definite z_0 at the moment $\delta\varrho/\varrho = 1$).

It is clear that measurements of the fluctuations of relic radiation allow a judgement of the time of galaxy formation only if radiation does not interact with matter after recombination. In fact, recombination may not occur instantaneously. Even if it occurred according to the equilibrium Saha equation, the results of Silk would require substantial corrections. Moreover, as shown by detailed calculations (Zeldovich *et al.*, 1968; Peebles, 1968), recombination of hydrogen does not occur according to the equilibrium Saha equation, but much more slowly. When a protogalaxy or proto-cluster of galaxies becomes transparent to radiation, the optical depth for Thompson scattering by matter between the proto-objects and the observer is still very large, as a result of which temperature fluctuations of relic radiation are smoothed out. Here by proto-object we mean a small perturbation of density at the moment of recombination enclosing a mass which later is converted into a presently existing object. The thesis about the small values of adiabatic fluctuations considered by Silk is confirmed by the spasmodic increases of the amplitudes of density perturbations during recombination.

Consideration of processes occurring before recombination leads to the following picture: at a later stage of expansion the amplitude of density perturbations turns out to be a periodic function of wavelength (mass). Such a picture was previously obtained by Sakharov (1965) for a cold model of the Universe. Together with density perturbations, peculiar plasma motions (connected in particular with these perturbations) occur superimposed on the general cosmological expansion. The value $\delta T/T$ mentioned above is the change of temperature measured by an observer moving together with the plasma: an observer on Earth also measures a change of intensity (fluctuation) due to the Doppler effect which equals $\delta T/T=(u/c)\cos\theta$, where u is the velocity of the plasma and θ is the angle between the velocity and the direction to the observer.

Throughout the entire article $\delta\varrho/\varrho$ will be the density perturbations and $\delta T/T$ the temperature fluctuations. Due to the long time of recombination adiabatic temperature perturbations are smoothed out and effects connected with the presence of peculiar velocities determine the observed temperature fluctuations of relic radiation connected with initial perturbations. Since the growth of adiabatic and entropy density perturbations follow the same law in the stage after recombination, their velocities of peculiar motions and temperature fluctuations of the radiation, which are related to various types of perturbations, turn out to be of the same order. Entropy perturbations played no role in Silk's approximation. It is quite probable that at the time of formation of present objects, i.e. when density perturbations became large $\delta\varrho/\varrho\geqslant 1$, a secondary heating and ionization of the gas occurred. Due to Thompson scattering, the ionized gas decreases the amplitude of fluctuations of radiation arriving from the period $z\sim 1000$. On the other hand for small z the ionized gas itself creates fluctuations of radiation due to both its macroscopic motion, and the simultaneous presence of inhomogeneous heating and thermal motion of electrons. In conclusion we compare the effects described above with the gravitational effects calculated by Sachs and Wolf (1967). Gravitational influence on radiation for the same values of $\delta\varrho/\varrho$ is much smaller than the effects described above, but it continues after matter becomes completely transparent to radiation. It is important for large masses.

2. Recombination and Its Influence on Density Perturbations and Temperature Fluctuations

In this section and future sections we will use the following notation: z=red shift; $\Omega=\varrho/\varrho_c=n/n_c$=present dimensionless average density of matter in the Universe, $\varrho_c=3\ H_0^2/8\ \pi G=2\times 10^{-29}$ g/cm^3, $n_c=10^{-5}$ cm^{-3}=critical density, $H_0=10^{-10}$ yrs^{-1}=Hubble constant, $T_0=2.7$ K=present temperature of relic radiation, $I=13.6$ eV=ionization potential of atomic hydrogen, and $w=8$ sec^{-1}=probability of a two quantum transition in a hydrogen atom from the 2s state to the ground state $\mathrm{H(2s)}\rightarrow\mathrm{H(1s)}+\gamma_1+\gamma_2$. The concentrations of corresponding particles are designated by the symbols p, e, H, and ϱ_r and ϱ_m are the energy densities of radiation and matter, respectively. For the present temperature of radiation $T_0=2.7$ K ϱ_m and ϱ_r are comparable in the process of expansion of the Universe for $z=4\times 10^4\ \Omega=z_1\Omega$. At the stage before recombination of the initial plasma the speed of sound depended on the relation between ϱ_r and ϱ_m,

$$a_s = \frac{c}{\sqrt{3}}\left(\frac{4\varrho_r}{4\varrho_r + 3\varrho_m}\right)^{1/2} = \frac{c}{\sqrt{3}}\left(1 + \frac{3}{4}\frac{\varrho_m}{\varrho_r}\right)^{-1/2} = \frac{c}{\sqrt{3(1 + \frac{3}{4}\Omega(z_1/z)}}.$$

A. RECOMBINATION RATE

Solving the equation of recombination (4) from the work of Zeldovich *et al.* (1968), it is easy to find that in the region $900<z<1500$,

$$x(z) = \frac{p}{p + \mathrm{H}} = \frac{e}{e + \mathrm{H}} = \frac{A}{z\sqrt{\Omega}}\, e^{-B/z},$$

where

$$A = \frac{(2\pi m_e k T_0)^{3/2}\, H_0 I}{4 w n_c k T_0 h^3} = 6 \times 10^6,$$

and

$$B = \frac{I_{23}}{kT_0} = \frac{I}{4kT_0} = 1.458 \times 10^4 .$$

This analytic solution is obtained from the following physical picture. For $z_r \sim 1500$ ($T_r = T_0 z = 4000$ K) in agreement with the Saha equation $p=e=\mathrm{H}$. In the process of further temperature decreases in the expansion of the Universe the Saha equation ceases to describe the process of recombination. Free electrons and radiation are only found in equilibrium with the excited levels of atomic hydrogen; the recombination rate is determined by two-quantum decays of the 2s level and is described by the Equation

$$\mathrm{d}p/\mathrm{d}t = -\,3Hp - w\mathrm{H}_{23}; \quad \mathrm{H}_{23} = \mathrm{const}\, p^2\, e^{-(B/z)} \tag{2}$$

the solution of which is given by Equation (1). For $z<900$ thermodynamic equilibrium is destroyed between H_{23} and free protons and electrons, the ionization time from the 2s levels by collisions with quanta and electrons becomes greater than w^{-1} and therefore the solution (1) is correct only in the interval $900<z<1500$. The optical depth of the Universe due to Thompson scattering equals

$$\tau(z) = \Omega^{1/2}\sigma_T n_c H_0^{-1} \int_0^z \eta^{1/2} x(\eta)\, \mathrm{d}\eta = \tau_0 + a z^{3/2}\, e^{-B/z}, \tag{3}$$

where $a = \sqrt{\Omega}\sigma_T n_c c H_0^{-1}\,(A/B) = 27.3$. The constant $\tau_0 = \sqrt{\Omega}\sigma_T n_c c H_0^{-1} \int_0^{900} x(z)\, z^{1/2}\, \mathrm{d}z = 0.4$ is easy to find with the equations for $x(z)$ for $z<900$, given in Zeldovich *et al.* (1968).

B. WEAKENING OF FLUCTUATIONS

We will find the moment z_2 when the proto-object which interests us with mass

$M = \frac{4}{3}\pi\varrho(z_2)\, l^3(z_2)$ becomes transparent (the condition of transparency: $\Omega\, \sigma_T l(z_2)\, n_c z_2^3 \sim 1$)

$$z_2 = \frac{4.4 \times 10^4}{7.5 + \ln(M\Omega^{1/2}/M_\odot)}. \tag{4}$$

Equation (4) is correct only for $M > 10^9\, M_\odot$ when $z < 1500$ and Equation (3) is applicable.

It is clear that the adiabatic relation $\delta T/T = \delta\varrho_m/3\, \varrho_m$ will not be satisfied and the temperature is smoothed out somewhat earlier than the moment z_2. Thus z_2 is larger, the smaller the mass M. On the other hand, at this moment the optical depth to a present observer is still large and adiabatic fluctuations of radiation connected with masses smaller than $2 \times 10^{15}\, \Omega^{-1/2}\, M_\odot$ will be weakened strongly after scattering by $e^{-\tau}$ times, where

$$\tau = 0.4 + 8.3 \times 10^4 (M\Omega^{1/2}/M_\odot) = 0.4 + (6 \times 10^{14}\, M_\odot/M\Omega^{1/2}). \tag{5}$$

For example, if the density of the Universe $\Omega = 0.1$, the effect from a single galaxy like ours with $M = 10^{11}\, M_\odot$ amounts to a 10^{11}-fold decrease. Below we will need the function

$$e^{-\tau}\frac{\mathrm{d}\tau}{\mathrm{d}z} = \sigma_T n_c c H_0^{-1} A z^{-1/2} \exp\left\{- a z^{3/2} e^{-B/z} - \frac{B}{z} - \tau_0\right\}, \tag{6}$$

which in agreement with (3) has a sharp maximum for $z_{\max} = 1055$ ($\mathrm{e}^{-\tau}(\mathrm{d}\tau/\mathrm{d}z)_{z=z[\max]} = 3.32 \times 10^{-3}$) and exponentially decreases in both directions, the value of the function decreasing to half its maximum value for $z_3 = 960$ and $z_4 = 1135$. It will be convenient in what follows to approximate this function by a Gaussian function with dispersion $\sigma_z = 75$ whose integral equals 1.

C. AMPLITUDE JUMP OF THE ADIABATIC PERTURBATIONS

In the process of recombination the amplitudes of the adiabatic density perturbations grow. Before recombination the Jeans wavelength was equal to

$$\lambda_\mathrm{J} = a_\mathrm{s}\left(\frac{\pi}{G(\varrho_r + \varrho_m)}\right)^{1/2} \approx \frac{c}{\sqrt{G}}\frac{\sqrt{\varrho_r}}{\varrho_r + \varrho_m} \approx \frac{c}{\sqrt{G\varrho_c}}\frac{1}{z\sqrt{z_1\Omega}\,(1 + (z/z_1\Omega))}.$$

We note that in a period $z > z_1\Omega\, (\varrho_r > \varrho_m)$ it will be of order ct. The Jeans mass of matter equalled

$$M_\mathrm{J} = \frac{\pi}{6}\varrho_m \lambda_\mathrm{J}^3 = \frac{10^{16}\, \Omega^{-2}}{(1 + (z/z_1\Omega))^3}\, M_\odot \tag{7}$$

i.e. it grew with decreasing z right up to $z_1\Omega$, but afterwards practically did not change. On a smaller scale density perturbations were manifested by sound waves of amplitude $\delta\varrho/\varrho$ which were connected with the velocity u of motion of matter by the relation

$$\left.\frac{\delta\varrho}{\varrho}\right|_{\mathrm{ion}} = \frac{K}{\omega}\bar{u}, \tag{8}$$

where **K** is the wave vector, $\omega = a_s K$, and the average is carried out over a half-cycle. Furthermore, $\delta\varrho/\varrho|_{\text{ion}}$ and $\delta\varrho/\varrho|_{\text{neutral}}$ are the corresponding density perturbations before and after time of recombination, respectively, and only close to the time of recombination. After recombination the Jeans wavelength which is determined only by the plasma pressure decreases ($m_J = 10^5 \div 10^6\ M_\odot$) and perturbations corresponding to masses greater than m_J grow as $t^{2/3}$. Using the equation of continuity (see below) it can be shown that

$$\left.\frac{\delta\varrho}{\varrho}\right|_{\text{neutral}} \approx \tfrac{3}{5} K t \bar{u}, \tag{9}$$

where t corresponds to the hydrodynamic time during recombination and the wave number $K = 10^3 z (H_0/c)\,(10^{15} \Omega M_\odot / M)^{1/3}$, $\delta\varrho/\varrho$ goes asymptotically to $\delta\varrho/\varrho|_{\text{neutral}}$ in the hydrodynamic time. Comparing (8) and (9) we see that the recombination of hydrogen in the Universe leads to an increased amplitude of density perturbations since the velocity of matter may not strongly change, as given by the equation

$$\frac{\left.\dfrac{\delta\varrho}{\varrho}\right|_{\text{neutral}}}{\left.\dfrac{\delta\varrho}{\varrho}\right|_{\text{ion}}} = \tfrac{3}{5} a_s K t \approx \left\{\frac{M_J(z_r)}{M}\right\}^{1/3} = \frac{1}{1 + 27\Omega}\left(\frac{2 \times 10^{20}\ \Omega}{M}\right)^{1/3}. \tag{10}$$

For $\Omega = 0.1$ and $M = 10^{11}\ M_\odot$ this relation is close to 2×10^2, which supports the thesis about the smallness of initial adiabatic temperature fluctuations of radiations $\delta T/T < (\delta\varrho/3\varrho)_{\text{neutral}}$. Equation (10) is true only for masses $M < M_J(z_r)$.

D. THE PERIODIC AMPLITUDE DEPENDENCE OF ADIABATIC DENSITY PERTURBATIONS AND TEMPERATURE FLUCTUATIONS

The picture presented above is only a rough approximation since the phase relations between density and velocity perturbations in standing waves in an ionized plasma were not considered. As mentioned in the introduction, Sakharov (1965) showed that the amplitude of perturbations of matter at a later stage when pressure does not play a role (in our case after recombination) turns out to be a periodic function of wave-length. This characteristic dependence is superimposed on the usually assumed power law dependence. From the expression for the dependence of the Jeans mass on the red shift we find z_J, the time when a given mass M is equal to the Jeans mass

$$z_J = z_1 \left[\left(\frac{10^{16}\ \Omega^{-2}}{M}\right)^{1/3} - 1\right] \Omega .$$

In the interval between z_J and z_r the adiabatic density perturbations of a given scale are manifested by standing sound waves. Waves corresponding to different scales have different frequencies and different periods of existence (see Figure 1). As a result, at the moment of recombination sound waves should have occurred with phases depending on the scale, i.e. density perturbations (and corresponding velocity perturbations and temperature fluctuations of the radiation) should depend on the mass of the

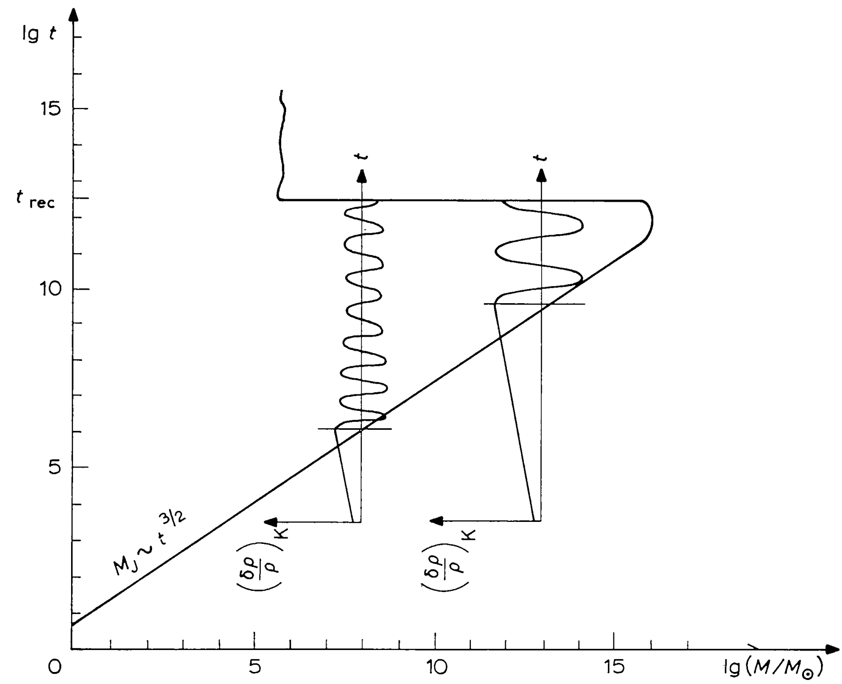

Fig. 1a. Diagram of gravitational instability in the 'big-bang' model. The region of instability is located to the right of the line $M_J(t)$; the region of stability to the left. The two additional lines of the graph demonstrate the temporal evolution of density perturbations of matter: growth until the moment when the considered mass is smaller than the Jeans mass and oscillations thereafter. It is apparent that at the moment of recombination perturbations corresponding to different masses correspond to different phases.

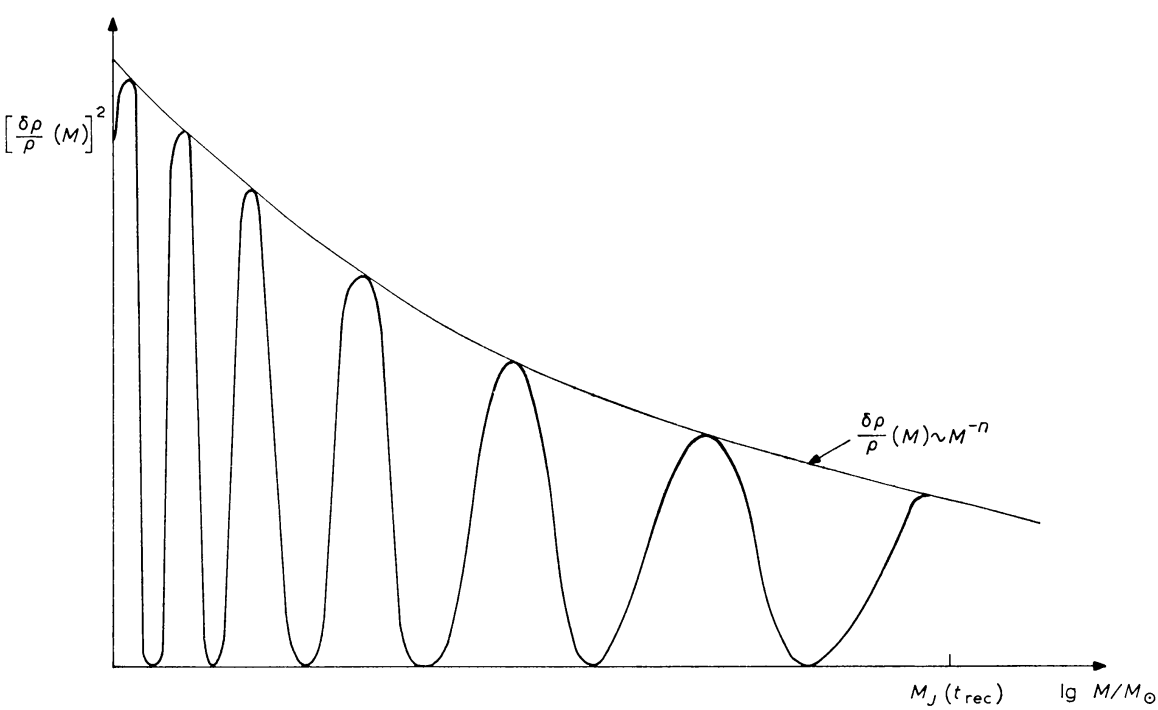

Fig. 1b. The dependence of the square of the amplitude of density perturbations of matter on scale. The fine line designates the usually assumed dependence $(\delta\varrho/\varrho)_M \sim M^{-n}$. It is apparent that fluctuations of relic radiation should depend on scale in a similar manner.

proto-object. From the previous result of the presence of growing modes, ion-acoustic waves at the stage $z_r < z < z_J$ should only be standing waves

$$\left(\frac{\delta\varrho}{\varrho}\right)_k \sim \sin\left(\int_{z_J}^{z} \omega \, dt + \varphi_k\right),$$
$$u_k(z) \sim \cos\left(\int_{z_J}^{z} \omega \, dt + \varphi_k\right) = \sin\left(\int_{z_J}^{z} \omega \, dt + \varphi_k + \frac{\pi}{2}\right). \tag{11}$$

The conditions for joining solutions at the moment z_J require equal phases for sound waves of all scales. We note that $(\omega t)_{z_J}$ does not depend on k since $\omega = a_s k$ and $a_s t \sim \lambda_J \sim k_J^{-1}$. Then at the moment of recombination z_r,

$$\left(\frac{\delta\varrho}{\varrho}\right)_k \sim \sin\varphi_r, \quad u_k \sim \cos\varphi_r .$$

Considering that $z_r \lesssim z_1 \Omega$ for all possible Ω's, we find*

$$\varphi_r = \int_{z_J}^{z_r} \omega \, dt + \varphi_0 \approx \frac{ck}{\sqrt{3 z_1 \Omega z_r} H_0} + \varphi_1 = bk + \varphi_1 ,$$

where

$$\varphi_1 = \varphi_0 - (\omega t)_{z_J} .$$

Now it is easy to find the zeros of the functions $\sin\varphi_r$ and $\delta\varrho/\varrho\,(M)$,

$$\varphi_r = n\pi, \quad n = 0, 1, 2, 3, \ldots$$
$$bk_0 + \varphi_1 = n\pi ,$$
$$k_0 = (n\pi - \varphi_1)/b ,$$

which corresponds to the Equations

$$(M_J/M_0)^{1/3} = n\pi - \varphi_1 ,$$
$$(M_0/M_J) = \frac{1}{(n\pi - \varphi_1)^3} \underset{n \gg 1}{\approx} \frac{1}{(n\pi)^3} . \tag{12}$$

The zeros of the function $u(M)$ will be moved by

$$\frac{M_0}{M_J} = \frac{1}{\{(n + \frac{1}{2})\,\pi - \varphi_1\}^3} . \tag{13}$$

It was shown in the first paragraph that the amplitude jump of density perturbations at the recombination time is connected with the presence of the velocity u so that the

* We replace $\int_{z_J}^{z_r} \omega \, dt$ by $\omega t|_{z_r} - \omega t|_{z_J}$ which considerably simplifies the calculation, but does not significantly change the qualitative results.

actual mass distribution of objects should be described by the last equation. It is barely possible to observe such a distribution since the mass of a galaxy $M \sim 10^{11}\, M_\odot$ for $\Omega = 1$ corresponds to $n \sim 15$, and for $\Omega = 0.1 - n \sim 70$ and $\Delta M/M \ll 1$. In the case $\Omega = 1$, this effect gives $\Delta M \sim M$ for a cluster of galaxies, but it is possible that it is masked by processes at a later stage of expansion of the Universe.

The dependence on the scale of both the amplitude of density perturbations and the speed of material motion should be reflected in the fluctuations of relic radiation. We are preparing to investigate this question elsewhere. We note that only observations of the small-scale fluctuations of relic radiation with a periodic dependence on scale may give information on the large-scale density perturbations which are small at the present time. The effect considered in the paragraph is applicable only to adiabatic density perturbations and will allow a choice to be made between density perturbations of various types.

3. The Observational Picture for Initial Fluctuations

In the first section of this article it was shown that adiabatic temperature fluctuations of small scale are strongly damped in the process of recombination. Below we shall consider fluctuations connected with the motion of matter at the time of recombination.

A. SCATTERING ON MOVING MATTER

With the aid of the equation of continuity it is possible to find the velocity of motion of matter for any given density perturbation

$$\frac{\partial \dfrac{\delta \varrho}{\varrho}}{\partial t} = \operatorname{div} u\,. \tag{14}$$

The scattering of quanta on moving electrons leads to a frequency shift (depending on the direction of motion) due to the Doppler effect. In the first approximation this effect is proportional to the optical depth of moving matter and decreases, as already mentioned in the first section, due to the subsequent scattering of radiation propagating from the proto-object to the observer, i.e.

$$\frac{\delta T}{T} = \int\limits_0^\infty \frac{u_1(z)}{c}\, e^{-\tau(z)} \frac{\mathrm{d}\tau}{\mathrm{d}z}\, \mathrm{d}z\,, \tag{15}$$

where u_1 is the projection of velocity along the direction of the ray and the integral is carried out along the ray $r = r(z)$, $t = t(z)$, $\theta = \text{const}$, and $\phi = \text{const}$. The function $e^{-\tau}(\mathrm{d}\tau/\mathrm{d}z)$ has already been found above (see Equation 6). All the effect (15) is concentrated in the region of the maximum of the function of Equation (6). The maximum effect produces objects for which the sign of u_1 does not change within the limits of a Gaussian curve. This result is applicable for perturbations containing a mass $M > M_{\max}$, where

$$M_{\max} = \frac{\Omega \varrho_c z_{\max}^3}{2} \left[c H_0^{-1} \int_{z_3}^{z_4} \frac{\mathrm{d}z}{z^{2.5} \Omega^{0.5}} \right]^3 = 7 \times 10^{14}\, \Omega^{-1/2} M_{\odot}\,. \tag{16}$$

For smaller objects it is necessary to take into account the change in sign of u_1.

The mean square temperature fluctuations which interest us will be calculated below. We represent the density fluctuations and the velocity of matter connected with them in the form of Fourier integrals

$$\begin{aligned} \frac{\delta\varrho}{\varrho} &= \frac{1}{(2\pi)^3} \int a_{\mathbf{p}}\, e^{i\mathbf{pr}}\, \mathrm{d}^3 p\,, \\ \frac{\mathbf{u}}{c} &= \frac{1}{(2\pi)^3} \int \mathbf{b}_{\mathbf{p}}\, e^{i\mathbf{pr}}\, \mathrm{d}^3 p\,, \end{aligned} \tag{17}$$

for which the dimensionless variable r is defined by

$$r = 1 - \frac{R H_0}{2c} = (1 + \Omega z)^{-1/2}\,, \tag{18}$$

where

$$R = \int_t^{t_0} \frac{\mathrm{d}x}{a(x)} = \frac{2c}{H_0} \left\{ 1 - \left(\frac{t}{t_0}\right)^{1/3} \right\} = \frac{2c}{H_0} \left\{ 1 - (1 + \Omega z)^{-1/2} \right\}$$

is the co-moving horizon coordinate and $a(t)$ is the radius of the Universe. With this definition of r,

$$p = \frac{K}{1 + z} \frac{2c}{H_0} = 2 \times 10^3 \left(\frac{M \Omega^{-1}}{10^{15}\, M_{\odot}} \right)^{-1/3} \tag{19}$$

does not depend on z. In (19), $M = \frac{4}{3}\pi (\pi/p)^3\, (2c/H_0)^3\, \varrho_c \Omega$ is the mass within a half-wavelength radius sphere. In agreement with Doroshkevich and Zeldovich (1963) after recombination perturbations vary with time according to the law

$$\frac{\delta\varrho}{\varrho}(t) = F \left[\left(\frac{t}{t_r}\right)^{2/3} - \left(\frac{t}{t_r}\right)^{-1} \right] \tag{20}$$

for any $z < z_r$ if $\Omega = 1$, and for $z > \Omega^{-1}$ if $\Omega < 1$. In Equation (20) it is assumed that the density perturbations $\delta\varrho/\varrho$ were small, for $t = t_r$, $(\delta\varrho/\varrho)\,(t_r) = 0$ before recombination and the perturbations were excited due to the presence of a velocity. The interesting mode for us is the growing mode for $t \gg t_r$,

$$\frac{\delta\varrho}{\varrho}(t) \sim t^{2/3} \sim (1 + z)^{-1}\,. \tag{21}$$

If the observed objects were formed when z_0 and $\delta\varrho/\varrho(z_0) \sim 1$, then it follows from

(21) that

$$\frac{\delta\varrho}{\varrho}(z) = \frac{1+z_0}{1+z}.$$

Since p does not depend on z, $a_{\mathbf{p}}$ varies in the same way, and

$$\left(\frac{\delta\varrho}{\varrho}\right)^2 = \frac{1}{(2\pi)^3}\int |a_{\mathbf{p}}|^2\, \mathrm{d}^3 p \quad \text{and} \quad a_{\mathbf{p}} \sim \frac{1+z_0}{1+z}. \tag{22}$$

Using (20) with the Equation of continuity (14) for $t=t_r$

$$\frac{\partial \dfrac{\delta\varrho}{\varrho}}{\partial t} = \frac{5}{3} F \frac{1}{t_r} = \frac{5}{3} F H_0 z_r^{3/2} \Omega^{1/2}$$

and taking into account the relation between the physical coordinate $x=ct$ and the variable r

$$\operatorname{div}\mathbf{u} = \frac{1}{(2\pi)^3}\int i\mathbf{p}\mathbf{b}_{\mathbf{p}}\, e^{i\mathbf{p}\mathbf{r}}\, \mathrm{d}^3 p\, \frac{1}{2}\frac{H_0}{c}\, z,$$

we find the desired relation between $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$

$$\mathbf{b}_{\mathbf{p}} = -\, 5 i \Omega^{1/2} z^{1/2}\, \frac{\mathbf{p}}{p^2}\, a_{\mathbf{p}}. \tag{23}$$

In this calculation it is assumed that the transition from adiabatic plasma oscillations connected with the radiation to the growth of perturbations occurs instantaneously: the corresponding time is small in comparison to the hydrodynamic time.

Temperature fluctuations are evidence by velocity perturbations in agreement with (15) from which we find with the aid of (17) for $\mathbf{u}$ the Equation

$$\frac{\delta T}{T} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3 p \int \frac{\mathbf{b}_{\mathbf{p}}\mathbf{r}}{r}\, e^{-\tau + i\mathbf{p}\mathbf{r}}\, \mathrm{d}\tau = \frac{1}{(2\pi)^3}\int c_{\mathbf{p}}\, \mathrm{d}^3 p. \tag{24}$$

The coefficients $c_{\mathbf{p}}$ are calculated in the following manner: substituting $e^{-\tau}(\mathrm{d}\tau/\mathrm{d}z)$ in the form of a Gaussian function with maximum at $r_{\max} = \Omega^{-1/2} z_{\max}^{-1/2}$ and dispersion σ_r, and taking into account $\sigma_r = \frac{1}{2}\sigma_z z_{\max} \Omega_{\max}^{-3/2}$, we obtain by standard methods from (22), (23), and (24)

$$\begin{aligned} c_{\mathbf{p}} &= 0.54\sqrt{3}\pi^{3/2}\Omega^{1/2}\sigma_z \cos\theta\, p^{-5/2} z_{\max}^{-1}(1+z_0) \\ &\quad \times \exp\left[-\frac{p^2\sigma_z^2\cos^2\theta}{8 z_{\max}^3 \Omega} + i\mathbf{p}\mathbf{r}_{\max}\right] a_{\mathbf{p}}(z_0) \\ &= C a_{\mathbf{p}}(z_0)\cos\theta\, p^{-3/2} \exp\left[-\frac{\alpha}{2}\cos^2\theta + i\mathbf{p}\mathbf{r}_{\max}\right], \end{aligned} \tag{25}$$

where

$$\alpha = p^2\sigma_z^2 / 4 z_{\max}^3 \Omega.$$

The mean square temperature fluctuations of the radiation are equal

$$\left(\frac{\delta T}{T}\right)^2 = \frac{1}{(2\pi)^6}\int c_{\mathbf{p}'}\,\mathrm{d}^3 p' \int c_{\mathbf{p}}\,\mathrm{d}^3 p\,. \tag{26}$$

Due to the orthogonality of waves with different $\mathbf{p}$ and $\mathbf{p}'$ and using (25), we obtain

$$\left(\frac{\delta T}{T}\right)^2 = \frac{C^2}{(2\pi)^2}\int \mathrm{d}p p^{-3}\int_0^1 \cos^2\theta\, e^{-\alpha\cos^2\theta}\,\mathrm{d}(\cos\theta)\,. \tag{27}$$

The integral over angles is easy to calculate: namely,

$$I = \int_0^1 \cos^2\theta\, \mathrm{e}^{-\alpha\cos^2\theta}\,\mathrm{d}(\cos\theta)$$

$$= \int_0^1 y^2\, e^{-\alpha y^2}\,\mathrm{d}y = \frac{1}{2\alpha}\left\{\sqrt{\frac{\pi}{4\alpha}}\,\Phi(\alpha\sqrt{2}) - e^{-\alpha}\right\},$$

where

$$\Phi(x) = \frac{2}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}\,\mathrm{d}t$$

is the probability integral. For large $\alpha > 1$ $I \approx \sqrt{\pi/4\alpha}\sqrt{\alpha}$; for small $\alpha \ll 1$, $I \simeq \frac{1}{3}$; and for $\alpha = 1$, $I = \frac{1}{2}$. We adopt the approximation

$$I^{1/2} = \frac{2\pi^{1/4}}{\sqrt{3\pi^{1/4} + 2\alpha^{3/4}}} \tag{28}$$

(in which we increase by a factor of 2 the asymptotic value of $\delta T/T$ for small and large masses) and note that $\alpha = 1$ for $p = 10^3\,\Omega^{1/2}$ and $M = 8\times 10^{15}\,\Omega^{-1/2}\,M_\odot$. Supposing that $a_{\mathbf{p}}$ has a maximum corresponding to the mass M, we finally obtain from (25), (27) and (28)

$$\sqrt{\left(\frac{\delta T}{T}\right)^2} = 2\times 10^{-5}\,\frac{\left(\dfrac{M\Omega^{1/2}}{10^{15}M_\odot}\right)^{1/3}}{1 + 2.5\left(\dfrac{10^{15}M_\odot}{M\Omega^{1/2}}\right)^{1/2}}\,(1 + z_0)\,. \tag{29}$$

The characteristic angular scale of fluctutaion is (Zeldovich and Novikov, 1967)

$$\theta = lH_0/2c\psi(z, \Omega),$$

where

$$\psi = \Omega^{-2}(1 + z)^{-2}\left[\Omega z + (\Omega - 2)(\sqrt{1 + \Omega z} - 1)\right].$$

For $z \gg \Omega^{-1}$ which applies in the case of interest to us we obtain, taking into account that

$$l = 2\pi c H_0^{-1} p^{-1}(1 + z)^{-1},$$

$$\theta = 2\pi\Omega/p \approx 10'(M\Omega^2/10^{15}\,M_\odot)^{1/3}\,. \tag{30}$$

B. DISCUSSION

We note especially that perturbations corresponding to small masses in comparison with $10^{15}\ M_\odot$ give quite a small contribution to $\delta T/T$; for example, for a single object with mass $M=10^{11}\ M_\odot$, in the case $\Omega=1$ and $(\delta\varrho/\varrho)=1$ for $z_0=2$ for a wave vector inclined at an angle of 45° to the direction of the observer, we obtain $\delta T/T=3\times10^{-8}$ with Equations (6), (9) and (15). After integrating over all angles, $\delta T/T=10^{-8}$. Perturbations effectively arise from a region whose absolute size (for $z_{\max}=1055$ and $\sigma_n=75$) is of order $L\sim10^{23}$ cm. A proto-object with mass $M=10^{11}\ M_\odot$ had a size $l\sim10^{21}$ cm at that time. If in the length L there are $N=L/l$ characteristic lengths and one object produces an effect $(\delta T/T)_1$, then it would turn out that the integrated effect should be simply $\delta T/T=\sqrt{N}(\delta T/T)_1$ according to the combination rule for random values. Why is the effect much smaller in the proposed calculations and why is the dependence on the sizes and mass of an object different? Physically, this signifies an assumption about the specific strength of anticorrelations between velocities of motion of matter in neighbouring regions of space. It is most probable that at a distance of the order of the size of an object the velocity changes sign. This result is formally connected with the fact that at a quite early stage, long before recombination, initial adiabatic perturbations are established by the spectral function $a_{\mathbf{p}}$. There is no basis for considering that this function leads to a constant value for all or even for small $\mathbf{p}$ (see a discussion about this point in Zeldovich and Novikov, 1967). The so-called 'natural' or 'random' distribution, for which the elementary law $\delta\varrho/\varrho\sim n^{-1/2}$ and $\Delta n=n^{1/2}$ (where n is the number of objects in a given volume) is correct, corresponds to $a_{\mathbf{p}}=\text{const}$. The power law $a(p)\sim p^m$ for $m>0$ corresponds to a large degree or order: for example, for $m=2$, $\Delta n\sim n^{1/6}$ (Zeldovich, 1965), i.e. an anticorrelation between fluctuations in neighbouring volumes exists. Apparently, this type of dependence of $a(\mathbf{p})$ as $p\to0$ actually occurs in nature.

4. Fluctuations of Radiation Connected with a Secondary Heating of Matter in the Universe

After recombination the hydrogen in the Universe should remain neutral. However, in our neighbourhood neutral intergalactic hydrogen is not detected right up to $z\sim2$. The possible variants are:

(a) The average density of matter in the Universe considerably exceeds the average density which went into galaxies which corresponds to $\Omega=\frac{1}{45}$. Sometimes for z_h between $z_r=1500$ and $z=2$ a secondary nonequilibrium heating and ionization of the intergalactic plasma occurred; the main part of the plasma remained distributed homogeneously and did not enter into the matter of galaxies. In this case the optical depth of the ionized gas for Thompson scattering,

$$\tau = \Omega\sigma_T n_c c H_0^{-1}\int_0^{z_h}\frac{1+z}{\sqrt{1+\Omega z}}\,\mathrm{d}z = 6.65\times10^{-2}\,\Omega\int_0^{z_h}\frac{1+z}{\sqrt{1+\Omega z}}\,\mathrm{d}z \tag{31}$$

may turn out to be sufficient for a strong decrease of the initial fluctuations of radiation. Due to the large energy losses of the ionized plasma by the inverse Compton effect of electrons on quanta of relic radiation, there is a small probability that heating occurred much earlier than $z \sim 10$ (Sunyaev, 1968; Doroshkevich and Sunyaev, 1969) which results in $\tau < 2$, i.e., this heating may not decrease the fluctuations of relic radiation resulting from perturbations for $z \sim 1000$ by more than an order of magnitude.

(b) Matter is mostly concentrated in galaxies whose average density is extremely small $\Omega \sim \frac{1}{45}$. Further, if matter was ionized before the formation of galaxies, τ should be extremely small $\tau \ll 1$.

(c) Intergalactic gas in clusters of galaxies determines the average density of matter in the Universe. Since neutral hydrogen is not founded in clusters of galaxies, its density $< 10^{-7}$ cm^{-3} (Allen, 1968) so that this gas should have a high temperature leading to $\tau < 2$, as in the case (a).

We return to fluctuations excited by the interaction with electrons at the stage $z < 10$. In Zeldovich and Sunyaev (1969) it was shown that the spectrum of relic radiation is changed by Compton scattering of hot electrons on the photons of this radiation leading to an intensity increase for $h\nu > 3.83\, kT$, but to an intensity decrease in the Rayleigh-Jeans region which is most convenient for observations. The effective temperature for $h\nu \ll kT_0$ is

$$T = T_0\, e^{-2y}, \tag{32}$$

where the parameter y is determined by

$$y = \int \frac{kT_e}{m_e c^2}\, d\tau\,. \tag{33}$$

It is obvious that in variant (a) the inhomogeneity of electron temperature and the inhomogeneity of plasma heating in various directions should lead to temperature fluctuations of the radiation with possible observational effects even for $\tau < 1$; the equations for the Rayleigh-Jeans region are true even for this condition. In variant (c) the determination of $\delta T/T$ in the direction of the nearest cluster of galaxies presents considerable interest. In agreement with (32) variations of the small-scale fluctuations of relic radiation are sensitive to extended high-temperature objects with small electron densities; with an increase of the plasma temperature the optical depth due to bremsstrahlung decreases, but the parameter y and $\delta T/T$ grow. The intensity of X-ray and radio emission are proportional to the square of the electron density, but y is proportional to the first power. A decrease in intensity of relic radiation in the Rayleigh-Jeans part of the spectrum close to X-ray sources would resolve the question about the radiation mechanism and allow a precise determination of the size of sources. Thus, for example, the presence of intergalactic gas with a temperature of 3×10^8 K and $n_e \approx 10^{-3}$ cm^{-3} in the Coma cluster of galaxies whose diameter is 10^{25} cm (Felten *et al.*, 1967) would lead to $\delta T/T = -2y = -10^{-3}$. The sign of the effect (a decrease of T) and large cluster sizes which allow the elimination of the contribution of the brightest radio sources gives rise to the possibility of verifying the interpretation of X-ray data.

Limits on the temperature of the intergalactic gas were obtained from X-ray measurements of the background emission for $\Omega \sim 1$; $T_e < 10^6 (1+z)$ K (Rees *et al.*, 1968; Veinshtein and Sunyaev, 1968). On the other hand, the existing data on the fluctuations of relic radiation which gives $\delta T/T < 10^{-3}$ on a scale 10′–15° (Wilkinson and Partridge, 1967; Conklin and Bracewell, 1967) indicate an absence of extended objects with temperatures exceeding 10^8 K in the investigated areas of the sky. The final estimate does not depend on z, since y is a function of the ratio $kT_e/m_e c^2$.

5. Conclusions

Due to a decrease of adiabatic fluctuations of radiation by subsequent scattering for $M < 10^{15}\ \Omega^{-1/2}\ M_\odot$ the initial fluctuations will be determined by the effect of scattering on moving matter and in the case of adiabatic and entropy perturbations will be equal to

$$\sqrt{\overline{\left(\frac{\delta T}{T}\right)^2}} = 10^{-5} \left(\frac{M\Omega^{1/2}}{10^{15}\ M_\odot}\right)^{5/6} (1 + z_0), \tag{34}$$

where z_0 corresponds to the moment when $\delta\varrho/\varrho \sim 1$. However, for $M > 10^{15}\ \Omega^{1/2}\ M_\odot$ and large Ω, even taking into account the jump in amplitude of density perturbations due to recombination, adiabatic fluctuations become of the same order as fluctuations connected with the motion of matter. This is easy to see from a comparison of Equations (9), (10) and (15). We introduce for comparison two equations for $M > 10^{15}\ \Omega^{-1/2}\ M_\odot$

(a) effects connected with motion:

$$\sqrt{\overline{\left(\frac{\delta T}{T}\right)^2}} = 2 \times 10^{-5} \left(\frac{M\Omega^{1/2}}{10^{15}\ M_\odot}\right)^{1/3} (1 + z_0); \tag{35}$$

(b) adiabatic fluctuations:

$$\sqrt{\overline{\left(\frac{\delta T}{T}\right)^2}} = 10^{-6} (1 + 27\Omega) \left(\frac{M\Omega^{-1}}{10^{15}\ M_\odot}\right)^{1/3} (1 + z_0). \tag{36}$$

The second equation is true only for masses less than $M_J(z_r)$ (i.e., less than the Jeans mass in the period before recombination). For $M > M_J$ again the effects of scattering on moving matter become most important; the velocity and this effect which is proportional to the velocity grow as $M^{1/3}$ at a time when adiabatic temperature fluctuations in this region no longer depend upon mass, as

$$\sqrt{\overline{\left(\frac{\delta T}{T}\right)^2}} = 2 \times 10^{-5} \left(\frac{M\Omega^{1/2}}{10^{15}\ M_\odot}\right)^{1/3} \frac{\delta_0 \varrho}{\varrho}, \tag{37}$$

where $\delta\varrho/\varrho < 1$ is the density perturbation corresponding to the present time (if $\Omega = 1$ or for $\Omega < 1$ at $z \sim \Omega^{-1}$) to objects with mass $M > M_J(z_r) = 10^{16}\,\Omega^{-2}\,M_\odot$. After recombination entropy density perturbations evolve in the same way as adiabatic perturbations. Temperature fluctuations connected with them are excited by scattering on moving matter and are described by Equation (22). Using the functional form of $e^{-\tau}(d\tau/dz)$ from Equation (6), it is easy to calculate temperature fluctuations of the relic radiation for any given distribution of velocities. Sachs and Wolf (1968) estimated the temperature fluctuations of the relic radiation for $\Omega = 1$ excited by the gravitational action of the nearest objects (gravitational effects on the early stage of expansion have little effect because of the growth of density perturbations)

$$\sqrt{\overline{\left(\frac{\delta T}{T}\right)^2}} = 2 \times 10^{-6} \left(\frac{M}{10^{15}\,M_\odot}\right)^{2/3} \frac{\delta_0 \varrho}{\varrho}. \tag{38}$$

Comparing (35), (36) and (38) we see that for $\Omega = 1$ fluctuations of the radiation formed during recombination exceed gravitational effects up to masses of the order of $10^{18}\,M_\odot$. For the case $\Omega = 1$ the existing experimental data $\delta T/T < 10^{-3}$ on a scale $\theta \sim 10'$–$15°$ (Wilkinson and Partridge, 1967; Conklin and Bracewell, 1967) contradict $\delta\varrho/\varrho \sim 1$ for masses $M > 10^{19}\,M_\odot$.

In Table I fluctuations of the radiation corresponding to the same density perturbation of matter, but due to various physical effects, are compared. In the fifth column the probable values of the temperature perturbations connected with the peculiar

TABLE I

Temperature fluctuations of relic radiation for density $\Omega = 1$

Mass of object in $M_\odot$	Exciting cause of fluctuations				Density perturbation of matter
	Adiabatic connection	Scattering on moving matter	Gravitational influence	Doppler velocity of observer	
10^{11}	10^{-13}	10^{-8}			$\delta\varrho/\varrho = 1$
10^{13}	5×10^{-7}	10^{-6}			for $z_0 = 2$
10^{15}	10^{-4}	3×10^{-5}			
10^{15}	3×10^{-6}	2×10^{-6}	2×10^{-7}	10^{-4}	$\delta\varrho/\varrho = 0.1$
10^{17}	2×10^{-5}	10^{-5}	4×10^{-6}	5×10^{-4}	for $z = 0$
10^{19}	2×10^{-5}	5×10^{-5}	10^{-4}	2×10^{-3}	
			$\Omega = 1/45$		
10^{11}	10^{-13}	4×10^{-8}			$\delta\varrho/\varrho = 1$
10^{13}	5×10^{-7}	2×10^{-6}			for $z_0 = 45$
10^{15}	2×10^{-4}	5×10^{-4}			
10^{15}	2×10^{-5}	5×10^{-5}			$\delta\varrho/\varrho = 0.1$
10^{17}	10^{-4}	2×10^{-4}			for $z = 0$
10^{19}	5×10^{-4}	10^{-3}			

motion of the observer (i.e., our own galaxy) relative to the relic radiation are given. This motion should manifest itself with a 24-hour period. Investigation of one plane (Wilkinson and Partridge, 1967) gave for this value $\delta T/T|_{24\mathrm{h}} \lesssim 10^{-3}$. It is clear from the table that the absence of observed motion and a 24-hour asymmetry provide the most stringent bounds for large-scale perturbations. It is necessary, however, to emphasize the specific character of this variation: we can in principle only obtain data on the velocity relative to the relic radiation for one object – our solar system. The possibility remains that in the average of the peculiar velocity $\bar{u}$ over all galaxies, randomly the velocity of our galaxy is less than u_0. The *a priori* probability of this is of the order $\alpha < (u_0/\bar{u})^2$ for the observations in one plane as it was carried out at the present time (two components of u_0). If the velocity remains less than u_0 as before for the measurements in two perpendicular planes (all three components) we obtain $\alpha < (u_0/\bar{u})^3$. For $u_0 = 200$ km sec^{-1} which gives $\alpha > 0.01$ we find $\bar{u} < 1000$ km sec^{-1} = 3×10^{-3} c which corresponds to the fifth column in the table.

The scattering of quanta of relic radiation on hot electrons in clusters of galaxies may lead to temperature fluctuations of relic radiation $\delta T/T \lesssim 10^{-4}$–$10^{-5}$ since the observed background X-ray radiation in galactic clusters is contradictory to the anamolously high temperatures of the gas in galactic clusters for any $z < 4$. Thus it is difficult to expect $\delta T/T > 10^{-5}$ from observed objects with mass $M < 10^{15}$ $M_\odot$. At the same time fluctuations of the background radiation due to the presence of discrete sources of radio emission exceed 10^{-4}–10^{-5} and complicate the detection of effects connected with the formation of observed objects (Longair and Sunyaev, 1969).

Acknowledgement

The authors would like to thank A. F. Illarionov for discussions.

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